An Analytical Model to Select the Fleet Size for MAST Systems

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Abstract— **The mobility allowance shuttle transit (MAST) system is a hybrid transit system in which vehicles are allowed to deviate from a fixed route to serve flexible demand. A mixed integer programming (MIP) formulation for the static scheduling problem of multi-vehicle MAST (m-MAST) system is proposed in this paper. Based on the MIP formulation, we analyze the impacts of time headways between consecutive transit vehicles on the performance of a two-vehicle MAST system. An analytical framework is then developed to model the performance of both one-vehicle and two-vehicle MAST systems, which is used to identify the critical demand level at which an increase of the fleet size from one to two vehicles would be appropriate. Finally, a sensitivity analysis is conducted to find out the impact of a key modeling parameter w_1 on the critical demand.**

I. INTRODUCTION

Public transit services are divided into two broad categories: fixed-route transit (FRT) and demand responsive transit (DRT). The FRT systems are thought to be cost-efficient because of their ride-sharing attribute and sufficient loading capacity. But they are considered by the general public to be inconvenient since the fixed stops and schedule are not able to meet individual passenger's desire. This inherent lack of flexibility is the most significant constraint of fixed-route transit. The DRT systems are much more flexible to offer door-to-door pick-up and drop-off services. They have been operated in quite a few cities and working as an effective type of flexible transit service especially within low-density residential areas, such as examples in Denver (CO), Raleigh (NC), Akron (OH), Tacoma (WA), Sarasota (FL), Portland (OR) and Winnipeg (Canada) [1]. However, the associated high cost prevents the DRT to be deployed as a general transit service. As a result they are largely limited to specialized operations such as shuttle service, cab and Dial-a-Ride services which are mandated under the Americans with Disabilities Act. Thus, transit agencies are faced with increasing demand for improved and extended DRT service.

The mobility allowance shuttle transit (MAST) is an innovative concept that combines the cost-efficient operability of traditional FRT with the flexibility of DRT systems. It allows transit vehicles to deviate from a fixed route consisting of a few mandatory checkpoints to serve on demand customers within a predetermined service area, and thus can be both affordable and convenient enough to attract the general public. For the MAST system, the fixed route can be either a loop or a line between two terminals. The checkpoints are usually located at major transfer stops or high demand zones and are relatively far from each other. A hard constraint of the MAST system is the scheduled departure time from checkpoints. The characteristics of the MAST system have in several cases efficiently responded to the needs and wants of both customers and transit agency. However, compared with the traditional FRT systems, their application in practice has been quite limited so far.

The design and operations of the MAST system has attracted considerable attention in recent years. Quadrifoglio et al. [2] evaluated the performance of MAST systems in terms of serving capability and longitudinal velocity. Their results indicate that some basic parameters are helpful in designing the MAST system such as slack time and headway. Quadrifoglio et al. later developed an insertion heuristic scheduling algorithm to address a large amount of demand dynamically [3]. In Quadrifoglio and Dessouky's work [4], they carried out a set of simulations to show the sensitivity analysis for the performance of the insertion heuristic algorithm and the capability of the system over different shapes of service area. In 2008, Zhao and Dessouky [5] studied the optimal service capacity for the MAST system. Although these studies investigated the design and operations of the MAST system from various aspects, they are all for the single-vehicle MAST system.

Since the MAST system is a special case of the pickup and delivery problem (PDP), it can be modeled as a mixed integer program (MIP). The PDP has been extensively studied. Cordeau introduced an MIP formulation of the multi-vehicle Dial-a-Ride Problem (DARP) [6]. He proposed a branch-and-cut algorithm using new valid inequalities for DARP. This multi-vehicle DARP MIP formulation is a good reference for the multi-vehicle MAST MIP formulation. Cordeau and Laporte gave a comprehensive review on PDP, in which different mathematical formulations and solution approaches were examined and compared [7]. Lu and Dessouky (2007) formulated the multi-vehicle PDP as an MIP and developed an exact branch-and-cut algorithm to optimally solve multi-vehicle PDP of up to 5 vehicles and 17 customers without clusters and 5 vehicles and 25 customers with clusters within a reasonable time [8]. In [9], Cortes et al. proposed an MIP formulation for the PDP with transfers. Berbeglia et al. reviewed the most recent literature on dynamic PDPs and provided a general framework for dynamic one-to-one PDPs [10]. Quadrifoglio et al. proposed an MIP formulation for the static scheduling problem of a single-vehicle MAST system and solved the problem by strengthening the formulation

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with logic cuts [11].

While the single-vehicle MAST system is suitable for small service areas and low demand, a larger fleet size would be needed to respond to heavier demand to provide a satisfactory level of service to its customers. A design and operational problem arises when transit agencies have to decide whether it is necessary to increase the fleet size and what conditions would justify doing so. A related problem has been addressed by Quadrifoglio and Li in which an analytical model is derived to identify the critical demand at which it would be appropriate to switch operating policy. [12]. A similar basic idea is adopted in this paper for identifying the demand level at which a fleet increase would be desired. So far no published work has yet dealt with the design and operations of the complex multi-vehicle MAST system. In this paper, after summarizing the Mixed Integer Programming formulation for the static scheduling of the multi-vehicle MAST system, the analytical modeling for critical demand is derived and tested by simulation in which the performance of the multi-vehicle MAST system is compared with its single-vehicle counterpart.

II. FORMULATION

The multi-vehicle MAST system considered consists of a set of vehicles with predefined schedules along a fixed-route of C checkpoints (i=1,2,...,C). These checkpoints include two terminals (i=1 and i=C) and the remaining C-2 intermediate checkpoints. A rectangular service area is considered in this study as shown in Fig. 1, where L is the distance between the two terminals and W/2 is the maximum allowable deviation distance on each side of the fixed-route. Vehicles perform R trips back and forth between the terminals.

In this study, the transit demand is defined by a set of requests. Each request consists of pick-up/drop-off locations and a ready time for pick-up. There are four possible types of customers requests, which are shown below:

- PD (Regular): pick-up and drop-off at a checkpoint
- PND (Hybrid): pick-up at a checkpoint and drop-off at a random point
- NPD (Hybrid): pick-up at a random point and drop-off at a checkpoint
- NPND (Random): pick-up and drop-off at random points

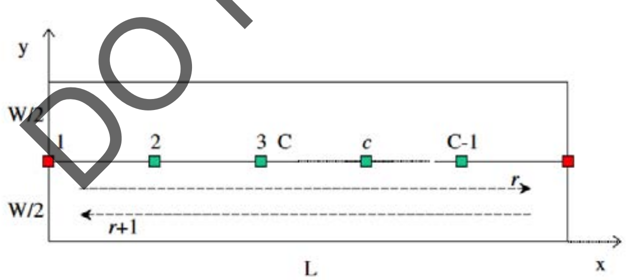

Fig. 1. MAST System

We consider the following two assumptions in formulating the multi-vehicle MAST problem: 1) the scenario is static and deterministic where the transit demand is known in advance; and 2) each request only has one customer and there is no capacity constraint for transit vehicles. The following presents the notations for the multi-vehicle MAST system:

Sets of Requests:

- $K_{PD}/K_{PND}/K_{NPD}/K_{NPND}$ = set of PD/PND/NPD/NPND requests
- $K_{HYB} = K_{PND} \cup K_{NPD}$ = set of hybrid requests (PND and NPD types)
- $ps(k) \in N$ = pick-up of k, $\forall k \in K \backslash K_{PND}$
- $ds(k) \in N$ = drop-off of k, $\forall k \in K \backslash K_{NPD}$
- $pc(k, r, v) \in N_0$ = collections of all the occurrences in the schedule (for each $r \in RD$ and each $v \in V$) of the pick-up checkpoint of k, $\forall k \in K_{PND}$
- $dc(k, r, v) \in N_0$ = collections of all the occurrences in the schedule (for each $r \in RD$ and each $v \in V$) of the drop-off checkpoint of k, $\forall k \in K_{NPD}$

Sets of Nodes:

- N_0 = checkpoints
- N_n = non-checkpoint stops
- $N = N_0 \cup N_n$

Sets of Arcs:

- A = all arcs

Sets of Trips:

- $RD = \{1, ..., R\}$ = set of trips
- $HYBR(k) \subset RD$ = feasible trips of k, $\forall k \in K_{HYB}$

Parameters:

- R = number of trips
- C = number of checkpoints
- V_e = number of vehicles
- V = set of vehicles
- $TC = [(C-1) \times R + 1] \times V_e$ = total number of stops at checkpoints in the schedule
- $TC_0 = (C-1) \times R + 1$ = number of checkpoint stops of one vehicle
- $TS = TC + |K_{PND}| + |K_{NPD}| + 2 \times |K_{NPND}|$ = total number of stops
- θ_i = scheduled departure time of checkpoint stop i, $\forall i \in N_0, (\theta_1 = 0)$
- τ_k = ready time of request k, $\forall k \in K$
- $\delta_{i,j}$ = rectilinear travel time between i and j, $\forall i, j \in N$
- b_i = service time for at stop i
- $w_1/w_2/w_3$ = objective function weights

Variables:

- $x^v_{i,j} = \{0, 1\}, \forall (i, j) \in A, \forall v \in V$ = binary variables indicating if an arc (i, j) is used by vehicle v ($x^v_{i,j} = 1$) or not ($x^v_{i,j} = 0$)
- t_i = departure time from stop i, $\forall i \in N$
- $\bar{t}_i$ = arrival time at stop i, $\forall i \in N \backslash \{1\}$
- p_k = pick-up time of request k, $\forall k \in K$
- d_k = drop-off time of request k, $\forall k \in K$
- $z^v_{k,r} = \{0, 1\}, \forall k \in K_{HYB}$ = binary variable indicating whether the checkpoint stop of the hybrid request k (a

pick-up if $k \in K_{PND}$ or a drop-off if $k \in K_{NPD}$) is scheduled in trip r of vehicle v, $\forall r \in RD, \forall v \in V$

The multi-vehicle MAST scheduling problem is formulated as the following mixed integer program (MIP):

$$\min \quad w_1 \sum_{v\in V} \sum_{(i,j)\in A} \delta_{ij} x^v_{i,j} + w_2 \sum_{k\in K} (d_k - p_k) + w_3 \sum_{k\in K} (p_k - \tau_k) \tag{1}$$

Subject to

$$\sum_{v\in V} \sum_{i} x^v_{i,j} = 1 \quad \forall j \in N\backslash\{1, TC_0+1, 2TC_0+1, ..., TC\}, \tag{2}$$

$$\sum_{v\in V} \sum_{j} x^v_{i,j} = 1 \quad \forall j \in N\backslash\{TC_0, 2TC_0, ..., TC\}, \tag{3}$$

$$\sum_{j} x^v_{i,j} = \sum_{i} x^v_{j,i} \quad \forall j \in N\backslash\{1, TC_0, TC_0+1, 2TC_0, ..., TC\}; v \in V \tag{4}$$

$$t_i = \theta_i \quad \forall i \in N_0 \tag{5}$$

$$p_k = t_{ps(k)} \quad \forall k \in K\backslash K_{PND} \tag{6}$$

$$d_k = \bar{t}_{ds(k)} \quad \forall k \in K\backslash K_{NPD} \tag{7}$$

$$\sum_{v\in V} \sum_{r\in HYBR(K)} z^v_{k,r} = 1 \quad \forall k \in K\backslash K_{HYB} \tag{8}$$

$$p_k \ge t_{pc(k,r,v)} - M(1 - z^v_{k,r}), \forall k \in K_{PND}, r \in RD, v \in V \tag{9}$$

$$p_k \le t_{pc(k,r,v)} + M(1 - z^v_{k,r}), \forall k \in K_{PND}, r \in RD, v \in V \tag{10}$$

$$d_k \ge \bar{t}_{dc(k,r,v)} - M(1 - z^v_{k,r}), \forall k \in K_{NPD}, r \in RD, v \in V \tag{11}$$

$$d_k \le \bar{t}_{dc(k,r,v)} + M(1 - z^v_{k,r}), \forall k \in K_{NPD}, r \in RD, v \in V \tag{12}$$

$$p_k \ge \tau_k \quad \forall k \in K \tag{13}$$

$$d_k \ge p_k \quad \forall k \in K \tag{14}$$

$$\bar{t}_j \ge t_i + \sum_{v\in V} x^v_{i,j} \delta_{i,j} - M(1 - \sum_{v\in V} x^v_{i,j}) \quad \forall (i,j) \in A \tag{15}$$

$$t_i \ge \bar{t}_i + b_i \quad \forall i \in N\backslash\{1, TC_0+1, ..., Ve \times TC_0 + 1\} \tag{16}$$

$$\sum_{j} x^v_{ps(k),j} - \sum_{j} x^v_{j,ds(k)} = 0 \quad \forall v \in V; k \in K_{PD} \cup K_{NPND} \tag{17}$$

$$\sum_{r\in HYBR(k)} \sum_{j} x^v_{pc(k,r,v),j} - \sum_{j} x^v_{j,ds(k)} = 0, \quad \forall v \in V; k \in K_{PND} \tag{18}$$

$$\sum_{j} x^v_{ps(k),j} - \sum_{r\in HYBR(k)} \sum_{j} x^v_{j,dc(k,r,v)} = 0, \quad \forall v \in V; k \in K_{NPD} \tag{19}$$

The objective function (1) minimizes the weighted sum of three different factors, namely the total vehicle time traveled, the total travel time of all passengers and the total waiting time of all passengers. Here waiting time is the time gap between the passengers ready time and the actual pick-up time. Network constraints (2), (3) and (4) allow each stop (except for the starting and ending nodes of each vehicle) to have exactly one incoming arc and one outgoing arc, which guarantee that each stop will be visited exactly once by the same vehicle. Constraint (5) forces the departure times from checkpoints to be fixed since they are pre-scheduled. Constraints (6) and (7) make the pick-up time of each request (except for the PND) and the drop-off time of each request (except for the NPD) equal to the departure time and the arrival time of its corresponding node, respectively. Constraint (8) allows exactly one z variable to be equal to 1 for each hybrid request, assuring that a unique ride of a unique vehicle will be selected for its pick-up or drop-off checkpoint. Constraints (9) and (10) fix the value of for each request depending on the z variable. Similarly, constraints (11) and (12) fix the value of variable for each request . Constraints (13) and (14) guarantee that the pick-up time of each passenger is no earlier than her/his ready time and is also no later than the corresponding drop-off time. Constraint (15) is an aggregate form of sub-tour elimination constraint similar to the Miller-Tucker-Zemlin (MTZ) constraint. Constraint (16) assures that at each node the departure time is no earlier than the arrival time plus the service time. Constraints (17), (18) and (19) are the key constraints in multi-vehicle MAST MIP formulation which assure that the pick-up and drop-off stop of each request are served by the same vehicle.

III. CRITICAL DEMAND

In this section, we derive the critical demand to identify the switch point between the single-vehicle MAST system and the multi-vehicle MAST system. The number of trips R and the number of checkpoints C are fixed for both MAST systems. The total demand (including all types of requests) is considered to be deterministic during the whole service period of MAST system. All the requests are assumed uniformly distributed in space and time, thus the non-checkpoint stops (NP and ND) are uniformly distributed in the service area. For simplicity, the time intervals between the departure times of two consecutive checkpoints are assumed to be uniform.

We define the following notation in this section:

- s_0 = service time at an inserted stop
- w = allowed deviation on the y-axis
- v = bus speed
- t = time interval between departure times of two consecutive checkpoints
- t_v = time headway between two consecutive vehicles
- $E(T^{PD}_{rd})$ = expected value of ride time of a PD
- $E(T^{PND}_{rd})$ = expected value of ride time of a PND
- $E(T^{NPD}_{rd})$ = expected value of ride time of an NPD
- $E(T^{NPND}_{rd})$ = expected value of ride time of an NPND

- $E(M)$ = expected value of travel miles of a vehicle
- $E(T_{rd})$ = expected value of ride time of a customer
- $E(T_{wt})$ = expected value of waiting time of a customer
- $\alpha, \beta, \gamma, \delta$ = portion of PD, PND, NPD, NPND requests respectively, and $\alpha + \beta + \gamma + \delta = 1$

A. Performance Measures and Utility Function

$E(M), E(T_{wt}), E(T_{rd})$ are the performance measures for the MAST system with associated weights. The weight assignment would change in different circumstances. A sensitivity experiment for w_1 is conducted in Section 4. We assume that the weight assignment is fixed for various cases here. The total utility value U is defined as following:

$$U = w_1 \times \frac{E(M)}{v} + w_2 \times E(T_{wt}) + w_3 \times E(T_{rd}) \quad (20)$$

This utility function is consistent with the objective function formulated in Section 2. It is obvious that lower values of total utility U indicate better performance of the MAST system. In the next subsections we will discuss the analytical computation of U in the one-vehicle case and the two-vehicle case, respectively, for the MAST operating policy. To calculate the expected values of the performance measures, we assume a static situation in which all the requests have been scheduled through a feasible and optimal procedure. This static situation can reflect an expected performance of the MAST system.

B. Analytical Modeling for the One-Vehicle Case

Since NP/ND customers are uniformly distributed within the whole service area, a service area delimited by any pair of consecutive checkpoints is defined as a basic unit. As depicted in Fig. 2, denote y as the vertical distance between any pair of NP/ND requests within the basic unit of service area, and we have the expected value of y: $E(y) = w/3$. Denote y' as the vertical distance between one of the two consecutive checkpoints (both located at w/2 on the y-axis) and its closest NP/ND stop within a basic unit of service area, and we have the expected value of y: $E(y') = w/4$. Then the formulation for three performance measures will be discussed.

1) Ride Time: Denote E_0^{PD} as the expected ride time of a PD customer within a basic unit of service area, n_0 as the demand density, meaning the average number of NP/ND stops that need to be inserted between two consecutive checkpoints in one trip, n' as the total number of NP/ND stops that need to be inserted into the schedule, N as the total number of customers. The following equations for n_0, n' and N hold:

$$n_0 = n'/[R(C-1)] \quad (21)$$

$$n' = |NPD| + |PND| + 2 \times |NPND| \quad (22)$$

$$N = |PD| + |NPD| + |PND| + |NPND| \quad (23)$$

Where an NPD (PND) request has one NP (ND) stop to be inserted, and an NPND request has two stops (one NP and one ND) to be inserted into the schedule. Then the formulation of E_0^{PD} is as following:

$$E_0^{PD} = \frac{L}{(C-1)v} + \frac{w}{v}[\frac{1}{4} \times 2 + \frac{1}{3}(n_0 - 1)] + s_0 \times n_0 \quad (24)$$

Where the first term is the travel time for horizontal distance between two consecutive checkpoints with no backtracking policy, the second term indicates the travel time for vertical deviation with n_0 stops scheduled, and the third term stands for the service time at n_0 stops. Extending to different units of service area, the expected ride time of a PD customer is in Equation (25).

$$E(T_{rd}^{PD}) = E_0^{PD} + (C-2)t/3 \quad (25)$$

Since all the requests are uniformly distributed, the NP (ND) stop of an NPD (PND) request is expected to be located at the middle of two consecutive checkpoints, which means the numbers of requests prior to and posterior to it within a basic unit of service area should be the same. Thus, the expected ride time of a PND or NPD customer whose pick-up or drop-off checkpoint is located within a basic unit of service area has the following equation:

$$E_0^{PND/NPD} = E_0^{PD}/2 \quad (26)$$

The expected ride time of a PND/NPD customer is half the value of a PD customer within one basic unit of service area. Similarly, considering the possibility of traversing different units of service area, the expected ride time of PND/NDP customer is:

$$E(T_{rd}^{PND/NPD}) = \frac{1}{2}E_0^{PD} + \frac{C-2}{3}t \quad (27)$$

Note that if the two non-checkpoint stops of an NPND request are scheduled within two consecutive checkpoints, the ride time of this NPND request is expected to be one third of the total average ride time between the two consecutive checkpoints (analogous to $E(|x-y|) = (U-L)/3$, if $x, y \in [L, U]$). Thus the expected ride time of an NPND customer with two stops scheduled within one basic unit of service area is given by Equation (28). The expected ride time of NPND customer is formulated in Equation (29).

$$E_0^{NPND} = E_0^{PD}/3 \quad (28)$$

$$E(T_{rd}^{NPND}) = \frac{E_0^{PD}}{3(C-1)} + \frac{C(C-2)}{3(C-1)}t \quad (29)$$

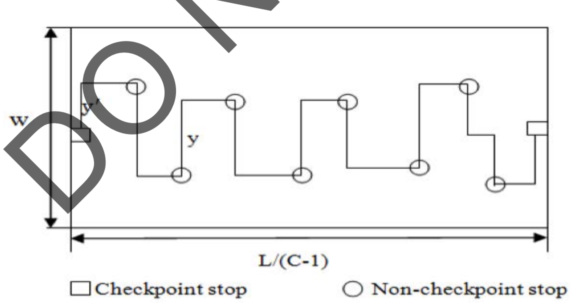

Fig. 2. Illustration for bus route within a basic unit area

Thus, the expected ride time of all the customers with different types of requests can be calculated by (30).

$$\begin{aligned} E(T_{rd}) &= E(T_{rd}^{PD}) \cdot |PD| + E(T_{rd}^{PND}) \cdot |PND| \\ &+ E(T_{rd}^{NPD}) \cdot |NPD| + E(T_{rd}^{NPND}) \cdot |NPND| \end{aligned} \tag{30}$$

2) Waiting Time: Since all the requests discussed here do not exceed the saturation demand and they are uniformly distributed without any obvious variation in this static situation for the analytical modeling, it can be concluded that the customer will be picked up within two trips (one cycle) of a vehicle for any type of request. So the expected waiting time of a customer with any type of request is equal to the total time of one trip. The following equation holds:

$$\begin{aligned} E(T_{wt}^{PD}) &= E(T_{wt}^{PND}) = E(T_{wt}^{NPD}) \\ &= E(T_{wt}^{NPND}) = (C-1)t \end{aligned} \tag{31}$$

Thus, we can get the expected value of waiting time of all the customers with different types of requests:

$$E(T_{wt}) = N(C-1)t \tag{32}$$

3) Miles Traveled: For the expected miles traveled by the vehicle during the whole service time, there are two terms formulated here. The first term $E(M_0)$ is the total horizontal miles that a vehicle has to travel. The second term $ext\_E(M)$ is the extra miles that a vehicle is supposed to travel due to the insertion of non-checkpoint stops. Thus the expected miles traveled by a vehicle during the whole service period is formulated as following:

$$\begin{aligned} E(M) &= E(M_0) + ext\_E(M) \\ &= R \cdot L + w[1/4 \times 2 + (n_0 - 1)/3]R(C-1) \end{aligned} \tag{33}$$

Combining the three performance measures, the utility function for one-vehicle case is:

$$\begin{aligned} U_1 &= \frac{w_1}{v}\{R \cdot L + \frac{w[R(C-1)}{6} + \frac{(\beta+\gamma+2\delta)N}{3}]\} \\ &+ w_2(C-1)tN + w_3N\{\frac{L}{(C-1)v} \\ &+ \frac{w}{v}[\frac{1}{4} \times 2 + \frac{1}{3}[\frac{(\beta+\gamma+2\delta)N}{R(C-1)} - 1]] \\ &+ s_0 \times \frac{(\beta+\gamma+2\delta)N}{R(C-1)}\}[\alpha + \frac{\beta+\gamma}{2} + \frac{\delta}{3(C-1)}] \\ &+ \frac{w_3N(C-2)t[\alpha+\beta+\gamma+C\delta/(C-1)]}{3} \end{aligned} \tag{34}$$

C. Analytical Modeling for the Two-Vehicle Case

For the two-vehicle MAST system, note that the average waiting time is determined by two extreme cases: the shortest waiting time (equal to 0) and the longest one. Also note that the system is symmetrical such that the case with time headway t_v is equivalent to the case with time headway $2(C-1)t - t_v$. Thus we have the following relationship for the expected waiting time:

$$\begin{aligned} E(T_{wt}^{PD}) &= E(T_{wt}^{PND}) = E(T_{wt}^{NPD}) = E(T_{wt}^{NPND}) \\ &= \begin{cases} (C-1)t - t_v/2, & for \quad t_v < (C-1)t \\ t_v/2, & for \quad (C-1)t \le t_v \le 2(C-1)t \end{cases} \end{aligned} \tag{35}$$

Apparently, in the range of [0, 2(C-1)t], the optimal t_v for (35) is $t_v = (C-1)t$ because of the symmetry of the system. In the following derivation it is assumed that $t_v = (C-1)t$, which means one vehicle starts from checkpoint 1 and the other one starts from checkpoint C simultaneously. Thus we have:

$$E(T_{wt}) = [N(C-1)t]/2 \tag{36}$$

Similar to the one-vehicle case, the expected miles traveled and the expected ride time for the two-vehicle case are formulated as following:

$$\begin{aligned} E(M) &= 2 \times [E(M_0) + ext\_E(M)] \\ &= 2\{R \cdot L + w[\frac{1}{4} \times 2 + \frac{1}{3}(\frac{n_0}{2} - 1)]R(C-1)\} \end{aligned} \tag{37}$$

$$E_0^{PD} = \frac{L}{(C-1)v} + \frac{w}{v}[\frac{1}{4} \times 2 + \frac{1}{3}(\frac{n_0}{2} - 1)] + s_0 \times \frac{n_0}{2} \tag{38}$$

$$E(T_{rd}^{PD}) = E_0^{PD} + (C-2)t/3 \tag{39}$$

$$E(T_{rd}^{PND/NPD}) = E_0^{PD}/2 + (C-2)t/3 \tag{40}$$

$$E(T_{rd}^{NPND}) = \frac{E_0^{PD}}{3(C-1)} + \frac{C(C-2)}{3(C-1)}t \tag{41}$$

Combining the three performance measures, the utility function for the two-vehicle case is:

$$\begin{aligned} U_2 &= \frac{2w_1}{v}\{R \cdot L + w[\frac{R(C-1)}{6} + \frac{(\beta+\gamma+2\delta)N}{3}]\} \\ &+ w_3N\{\frac{L}{(C-1)v} + \frac{w}{v}[\frac{1}{2} + \frac{1}{3}[\frac{(\beta+\gamma+2\delta)N}{2R(C-1)} - 1]] \\ &+ s_0 \times \frac{(\beta+\gamma+2\delta)N}{2R(C-1)}\}[\alpha + (\beta+\gamma)/2 + \frac{\delta}{3(C-1)}] \\ &+ \frac{w_3N(C-2)t[\alpha+\beta+\gamma+C\delta/(C-1)]}{3} \\ &+ \frac{w_2(C-1)tN}{2} \end{aligned} \tag{42}$$

D. Critical Demand

The utility functions for the one-vehicle and two-vehicle cases are derived and shown in Equations (34) and (42), respectively. By equating these two utility functions and solving for N, the critical demand N_c can be obtained. At this critical demand, the one-vehicle and two-vehicle systems will have the same system performance (including both operation cost based on vehicle miles traveled and service quality provided to customers). In other words, transit demand beyond this critical demand point would necessitate an increase in the fleet size.

By equating the two utility functions, the following quadratic equation can be obtained:

$$A_1N^2 + A_2N + A_3 = 0 \tag{43}$$

where,

$$\begin{aligned} A_1 &= \frac{w_3(\beta+\gamma+2\delta)}{R(C-1)}(\frac{w}{6v} + \frac{s_0}{2}) \\ &\cdot [1 - \delta - \frac{\beta+\gamma}{2} + \frac{\delta}{3(C-1)}] \end{aligned} \tag{44}$$

$$A_2 = \frac{w_2 t(C-1)}{2} \quad (45)$$

$$A_3 = -\frac{w_1}{v}[RL + \frac{wR(C-1)}{6}] \quad (46)$$

The critical demand can be obtained by solving the quadratic equation.

IV. EXPERIMENTS

In this section we conduct two types of experiments. First, we analytically derive the critical demand for switching between the one-vehicle and two-vehicle MAST systems and conduct numerical analysis. We also find the optimization results for the formulated MIP model using CPLEX. The optimization results confirm the derived critical demand. Second, we perform a sensitivity analysis for the weight of vehicle miles traveled.

All the runs are conducted using CPLEX 12.0 x64 with default settings using a desktop computer with Core 2 CPU @3.00 GHz and 8GB RAM. Table 1 summarizes the basic model input parameters.

As mentioned before, here L denotes the distance between the two terminals, W denotes the maximum allowable deviation distance on the y-axis, C denotes the number of checkpoints, R denotes the number of trips, $\delta_{s,s+1}$ denotes the rectilinear travel time between two consecutive checkpoints, b_s denotes the service time for boarding and disembarking at each stop, t denotes the time interval between departure times of two consecutive checkpoints, and w_1, w_2, w_3 are the objective function weights.

A. Validation of the Analytical Model

For various situations with different number of customers N, based on the derived utility functions from Section 3 and the given model input parameters, the analytical utility results for the one-vehicle case (ANA-1) and two-vehicle case (ANA-2) are calculated and shown in Table 2. The optimization results from the MIP model are also listed in Table 2. These results are approximated by two quadratic trend lines for both the one-vehicle case (Poly.(SIM-1)) and two-vehicle case (Poly.(SIM-2)) and are plotted in Fig. 3, which also includes two lines representing the analytical results.

From Fig. 3 the following observations can be made with regards to the utility function curves for the one-vehicle and two-vehicle MAST cases.

TABLE I
SYSTEM PARAMETERS

L	10 miles
W	1 mile
C	3
R	6
$\delta_{s,s+1}(s = 1, ..., TC-1)$	12 min
$b_s(s = 1, ..., TS)$	18 sec
$w_1/w_2/w_3$	0.4/0.4/0.2
t	25 min

TABLE II
UTILITY VALUES FROM ANALYTICAL RESULTS AND CPLEX RESULTS

N	Analytical Model		CPLEX	
	One-Vehicle	Two-Vehicle	One-Vehicle	Two-Vehicle
8	192.3	211.2	194.9	216.1
10	225.2	233.8	228.8	246.6
12	258.3	256.5	252.9	255.6
14	291.6	279.2	304.2	268.0
16	327.5	304.6	322.8	305.3
18	361.1	327.5	369.2	333.2
20	394.8	350.5	409.3	354.1

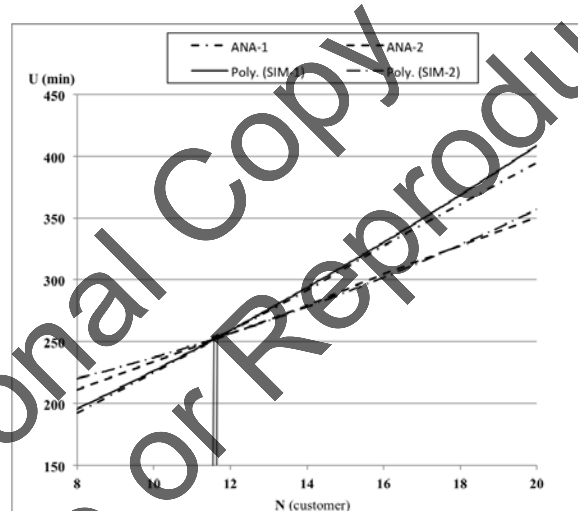

Fig. 3. Utility function curves for one-vehicle case and two-vehicle case

- The analytical results match the CPLEX results well for both cases even though there still exist some small deviations (e.g., when N is above 18 in the one-vehicle case). The analytical results are a little smaller than the corresponding CPLEX results, which might be caused by some idealized considerations of the analytical modeling that overestimate the system performance.
- The critical demand (the intersection point) at which the one-vehicle case and the two-vehicle case have the same utility function value is around 12, corresponding to the critical demand density $n_0 = 1$ (see (21) for definition of n_0). Below this critical demand value, applying the one-vehicle MAST system can result in lower utility function value (better performance). Beyond this critical demand point, the two-vehicle MAST system is preferable.
- In general, for each case the CPLEX result curve fits the analytical result curve very well. This suggests that both the analytical model can be used to estimate the actual utility function values and identify the critical demand.

B. Sensitivity Analysis

Special attention is paid to w_1, which reflects the weight of cost increase when another vehicle is introduced. To see how the critical demand N_c varies as a result of changing w_1, we set $w_2 : w_3 = 1 : 2$, $w_1 + w_2 + w_3 = 1$, and increase

TABLE III

N_c FOR VARIOUS w_1

	$w_1 = 0.25$	$w_1 = 0.4$	$w_1 = 0.5$
MIP	6.52	11.56	15.58
Analytical	5.88	11.64	17.28

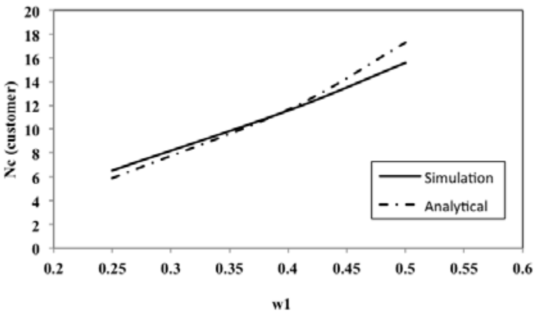

Fig. 4. N_c with various w_1 (CPLEX vs. analytical results)

w_1 from 0.25 to 0.5. The results are shown in Table 3 and Fig. 4.

Fig. 4 clearly indicate that N_c gets larger with the increase of w_1. In other words, if we put more weight on the total miles traveled, the critical demand to switch from the one-vehicle MAST system to its two-vehicle counterpart will also increase. This is expected because when switching from the one-vehicle system to two-vehicle system, the last two terms in the utility function (20) reflecting the service quality are significantly decreased, whereas the first term is nearly doubled. Thus, the changes in w_1 will affect this trade-off and lead to the increasing trend of critical demand as depicted in Fig. 4.

V. CONCLUSIONS

In this paper we propose an MIP for the scheduling of m-MAST and an analytical modeling framework to help MAST operators to identify the critical demand (N_c), which is used to decide when to switch from 1-MAST to 2-MAST. Utilizing this analytical model and the MIP formulation, we also compare the utility function values generated by the two methods for the 1-MAST system and the 2-MAST system. Finally, a sensitivity analysis is conducted to find out the impact of a key modeling parameter w_1 on the critical demand.

Experiments are conducted to find out the critical demand for switching between the 1-MAST and 2-MAST. The results show that the MIP and the analytical model generate approximately the same utility function values. This reasonable match demonstrates the validation and the effectiveness of the proposed analytical framework for critical demand modeling.

Since the MAST problem is NP-hard, the MIP can only optimally solve problems of moderate size. Future work will include the development of valid inequalities and logic constraints to strengthen the formulation and heuristic algorithms to allow the problems to be solved in real time and at large size. It would also be interesting to extend the analytical model to different MAST configurations (e.g., 3+ vehicles) and to identify the optimal fleet size as a function of demand.

VI. ACKNOWLEDGMENTS

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