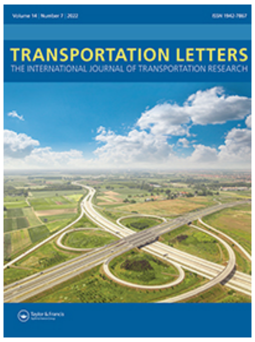

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Transportation Letters

The International Journal of Transportation Research

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/ytrl20

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To cite this article: Wei Lu, Luca Quadrifoglio, Dahye Lee & Xiaosi Zeng (2022): The ridesharing problem without predetermined drivers and riders: formulation and heuristic, Transportation Letters, DOI: 10.1080/19427867.2022.2116674

To link to this article: https://doi.org/10.1080/19427867.2022.2116674

Published online: 29 Aug 2022.

TRANSPORTATION LETTERS
https://doi.org/10.1080/19427867.2022.2116674

The ridesharing problem without predetermined drivers and riders: formulation and heuristic

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ABSTRACT

We consider a ridesharing service in which no driver and rider's roles are pre-determined, but left to decide by the system to further reduce costs compared to the typical version with preassigned roles. Travelers are motivated to participate in the service by saving individual transportation costs and accept its rules. We first formally define it as a generalized ridesharing optimization problem (RSP), propose its transformation into a single-depot multiple traveling salesman problem with pickup and delivery constraints (SDMTSP-PD) and provide its mixed-integer program (MIP) formulation. We then develop a polynomial-time solution method based on optimal pair matching among participants, improved by a construction insertion-based heuristic to obtain approximate solutions to the SDMTSP-PD. Experiments show that this approach could solve the problem very fast and provide near-optimal solutions and that the proposed RSP model provides substantial system-wide travel cost saving (25%+) and vehicle-trip saving (50%) compared to the non-ridesharing system and perform better than companion services with preassigned roles (P-RSP).

KEYWORDS
Ridesharing; driver; matching; insertion

Introduction and literature review

Ridesharing, by definition, occurs when travelers are willing to share both a private vehicle and the associated travel cost with others that have similar itineraries and compatible time schedules. The aim of ridesharing is to improve the efficiency of transportation by combining the fast travel time and convenience of private cars and the cost-efficiency of fixed-route transit to provide an attractive and viable alternative. Ridesharing arises as a desirable urban transportation option in the context of finite oil supplies, rising gas prices, never-ending traffic congestion, and environmental concerns. The private car occupancy rates are low: according to the recent reports (Federal Highway Administration 2017; Department for Transport Statistics 2020), the average passenger car occupancies in 2017 in the UK and the US were 1.6 and 1.67, respectively, meaning the vast majority of the trips are transporting 'empty seats.' The low occupancy, together with the large demand for automobile transportation at the peak-hours, leads to traffic congestion in many urban areas. According to the report published by the Texas A&M Transportation Institute (Schrank, Eisele, and Lomax 2019), the economic loss associated with congestion was 179 billion dollars in 2017 and is expected to grow to 237 billion dollars in 2025. Besides, private automobiles are also a major source of fuel consumption and carbon dioxide emissions, which contribute to air pollution and climate change. The overall motivation for developing and promoting ridesharing services is, therefore, to utilize the unused capacity of vehicles by allowing multiple riders on board to minimize the total vehicle miles traveled and improve system-wide indicators, such as congestion and related emissions.

Effective usage of ridesharing can potentially increase the occupancy rates (Morency 2007; Tafreshian, Masoud, and Yin 2020), reduce the number of single-occupancy vehicles; and thus reduce congestion and need for parking space, especially in large metropolitan areas and during peak hours (Morency 2007; Agatz et al. 2010; Stiglic et al. 2015, 2016; Tafreshian, Masoud, and Yin 2020). Moreover, the ridesharing system has 'scale effects.' As shown by Dailey, Loseff, and Meyers (1999), the relationship between the number of ridesharing participants and the number of carpools formed is quadratic. It means that ridesharing significantly affects traffic demand management (TDM) if large segments of the population are attracted to the service. Personal benefits, such as associated expenses shared among the participants and time saving using express or High Occupancy Vehicle (HOV) lanes (Chan and Shaheen 2012; Lee and Savelsbergh 2015), are expected. Moreover, the economic benefits associated with ridesharing, in turn, attract more travelers to participate in ridesharing services, thereby improving the utilization of transportation infrastructure capacity. Market leader Uber is currently shifting its focus to ridesharing services in major markets. Recent data shows that half of all Uber rides in San Francisco are for UberPool, which is Uber's ridesharing service (Schwieterman and Smith 2018). Other players in the market, such as Lyft and Ziro, share the same trend, as the popularity of smartphones makes efficient sharing and real-time communication possible and easy among all passengers.

Although ridesharing as a transportation solution is promising in a lot of ways, the operations of ridesharing systems introduce new challenges to both industry and academia, especially when considering spatio-temporal constraints from both drivers and riders (Martins et al. 2021). Ridesharing services have attracted growing attention from academia. See Agatz et al. (2012), Furuhata et al. (2013) and Tafreshian, Masoud, and Yin (2020) for a review on this topic. Baldacci, Maniezzo, and Mingozzi (2004) studied the carpooling problem in which travelers (mainly employees of a large employer) share rides to and from work, presenting an optimization problem and an exact method based on Lagrangian column generation. In contrast, Calvo et al. (2004) studied the same problem proposing a method with path construction and local search, which is formulated as a heuristic algorithm.

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In ridesharing's simplest form, a vehicle transports a single rider, and riders are not allowed to transfer between vehicles, which is called one-to-one matching (Tafreshian, Masoud, and Yin 2020). One-to-many forms of ridesharing problems indicate that drivers can serve multiple riders while the riders are not allowed to transfer between vehicles. Herbawi and Weber (2012) modeled a one-to-many dynamic ride-matching problem in which participants show up on short notice and proposed genetic and insertion heuristic algorithms. Di Febbraro, Gattorna, and Sacco (2013) modeled the one-to-many dynamic ridesharing systems as a mixed-integer optimization problem whose goal is to minimize the difference between the desired departure and arrival times. Stiglic et al. (2015) designed an algorithm to estimate the benefits of meeting points based on an extension of the traditional bipartite matching formulation. They discovered that introducing meeting points could significantly increase the capacity of ridesharing services and system-wide driving distance savings. Masoud and Jayakrishnan (2017a) relaxed the assumption of ridesharing services and proposed a binary program in a time-expanded network. Bei and Zhang (2018) study the ridesharing assignment problem from an algorithmic perspective. There are two distinct sets of riders and drivers in their modeling framework, and no time window constraints are considered. They specifically model the problem of assigning two riders to one driver as a combinatorial optimization problem, show its NP-hardness and devise an approximation algorithm with guaranteed performance bound. Bian and Liu (2019) designed a mechanism for passengers that use a first-mile ridesharing service given their personalized requirements. Li and Chung (2020) proposed a static mathematical model to increase the matching rate under travel time uncertainty and developed an extended insertion and tabu search heuristic algorithm.

The ridesharing problems are all closely related to the pickup and delivery problem (PDP, see Savelsbergh and Sol (1995) for a review). Many of the studies involved integer programming-based exact algorithms. Sexton and Bodin (1985) reported an exact algorithm based on Bender's decomposition. Lu and Dessouky (2004) developed a MIP formulation for the multiple-vehicle PDP. New valid inequalities were utilized to develop a branch-and-cut algorithm to solve the problem optimally. Quadrifoglio, Dessouky, and Ord´on˜ez (2008) developed a MIP formulation for the static single-vehicle Mobility Allowance Shuttle Transit (MAST) system, which is a variant of the PDP system. Logic cuts were proposed by the authors to strengthen the formulation and solve the problem. Cortes, Matamala, and Contardo (2010) presented a strict MIP formulation for the PDP and allowed passengers to transfer. Baldacci, Bartolini, and Mingozzi (2011) proposed an exact method for the PDP with time windows with a branch-and-cut-and-price algorithm. (Psaraftis 1980) developed dynamic programming techniques to solve the Dial-a-Ride Problem (DARP) and DARP with time windows later (Psaraftis 1983). Due to the fact that PDP is NP-hard (Lenstra and Kan 1981), besides exact solution methods, the research community has been focusing on heuristic approaches that can solve large instances of PDP in polynomial time while maintaining the quality of the solution. Jaw et al. (1986) first adapted the traditional insertion approach to solve the multi-vehicle dial-a-ride problem with time windows. The insertion-based constructive heuristic has then been developed to solve the PDP with time windows (Lu and Dessouky 2006), the single-vehicle MAST scheduling problem (Quadrifoglio, Dessouky, and Palmer 2007), which is a variant of the PDP, and the multiple-vehicle MAST problem (Lu, Lu, and Quadrifoglio 2011). Berbeglia, Cordeau, and Laporte (2010) gave a comprehensive review on dynamic PDP and discussed solution strategies. In a recent study, Hou, Li, and Zhang (2018) considered a ride-matching problem and a vehicle routing problem (VRP) simultaneously to maximize the average loading ratio of the entire system. In a broader range, (Vidal et al. 2013) provides a review of heuristics and meta-heuristics applied to VRPs with various characteristics and constraints. (Vidal et al. 2014) introduced a unified metaheuristic solver for VRPs with variants. Given the proposed solver matched or outperformed the existing algorithms for 29 VRP variants, the authors suggested that generality does not necessarily degrade performance for the problem classes of VRP with multiple attributes. Later, (Vidal, Laporte, and Matl 2020) contains a comprehensive review of recent trends in study objectives of VRPs and their integration with other research areas.

Some studies addressed ridesharing problems where the participants' role is not fully predetermined. Agatz et al. (2011) considered role assignment problem using a general graph matching model, which can be solved in polynomial time for a one-way trip matching problem. Lloret-Batlle, Masoud, and Nam (2017) proposed a matching and pricing mechanism for a peer-to-peer ridesharing system based on the Vickrey-Clarke-Groves (VCG) model. Masoud and Jayakrishnan (2017b) relaxed the assumption of riders and drivers forming mutually exclusive sets, instead, considered overlapping sets of drivers and riders, which maximizes the number of served riders in the system. Tafreshian and Masoud (2020) proposed a near-optimal solution found by applying a heuristic graph partitioning method. However, we are not aware of any work addressing and modeling the ridesharing optimization problem with undetermined drivers's role from the most general perspective, where willing participants are all equal and allowed to be either a driver or a rider. Roles are assigned by the system and optimized along with the associated routing problem. This novel feature allows for larger feasible regions and almost certainly better system-wide solutions, but also increases the complexity of the solution procedure, as new binary decision variables are needed and introduced to assign the participants' roles. Our work's aim and scope are to be among the first researchers to propose and model this novel and more general ridesharing service configuration, provide its formulation and suggest a solution approach. This paper is partially inspired by the modeling features of Bei and Zhang (2018); however, it fundamentally differs from it, causing it to become a special case of our model. Our solution approach is built on the optimal matching problem, proposed by Wang (2013) and solved in polynomial time for up to parties of two, but we improved it to provide solutions for larger ridesharing parties. This paper is also complemented by Lu and Quadrifoglio (2019), which identified a fair cost allocation scheme by finding the nucleolus of the game-theoretical framework built-in the ridesharing problem.

The highlights of our work are depicted in the remainder of this paper, which is organized as follows. Section 2 formally defines the general Ridesharing Optimization Problem (RSP). Section 3 develops a modeling transformation to generate an equivalent mixed-integer program based on the RSP. Section 4 introduces a solution procedure based on optimal matching and a construction heuristic to solve the RSP quickly with acceptable approximation. In Section 5, experiments are conducted to evaluate the quality of the developed model and algorithm, in addition to some statistical tests and sensitivity analyses. Finally, conclusions and future research direction are presented in Section 6.

The ridesharing optimization problem

We study the most generalized setting of the ridesharing problem. We consider a set of time-compatible travelers, with their origins/destinations, willing to participate in the ridesharing 'game' as riders or able to serve as designated drivers in a large scale ridesharing system. We aim to simultaneously make decisions on driver/rider role assignment, customer partition and route planning, with the goal of minimizing the system-wide total cost. Basically, for any given set of participants, the typical ridesharing service (with predetermined drivers) is one of the many instances of the generalized problem. Each instance is represented by a unique set of decision variables representing how many (and which) participants will serve as drivers. Each instance will then include a ridesharing plan with riders and routing. The best instance is the solution to the proposed generalized ridesharing problem. Time compatibility between riders indicates that ride times are similar and that potential time windows are large enough to be redundant constraints. We consider these assumptions realistic and acceptable when there is a large set of potential riders, a subset of which is preselected to be compatible with these assumptions. Practically speaking, in this paper, we are assuming that this preselection has already been performed. Furthermore, the proposed formulation has a robust structure that can be easily divided into clusters including riders sharing similar time schedules and solved in a parallel algorithm fashion.

Ridesharing system objectives

The objective of the ridesharing centralized system's decision-maker is to minimize the system-wide cost, which is mainly composed of and directly proportional to the total mileage traveled by all users. This objective is meaningful from a social perspective because total vehicle mileage is critically related to emissions of air pollutants and road congestion. Note that this objective is also closely related to minimizing total travel costs, or alternatively speaking, maximizing total travel cost savings, which is the direct motivation of ridesharing participants. We will later show that this system-wide objective has an alignment with individual participants' interests – minimizing personal travel costs.

Problem definition

Let P be the set of participants in a ridesharing system. Each participant $i \in P$ wants to travel from his/her origin s_i to his/her destination t_i. The set of origins and destinations are defined as $V_s = \{s_i \mid i \in P\}$ and $V_t = \{t_i \mid i \in P\}$, respectively, and the entire location set is $V = V_s \cup V_t$. Let $A = V \times V$ denote the edge set connecting all the vertices in V and $C \in \mathcal{R}^{|V| \times |V|}$ denote the cost matrix with c_{ij} representing traveling cost from location i to j. Then, we have a complete graph $G = (V, A)$ and its edge cost matrix C as input. To formally introduce the ridesharing optimization problem, we first present some definitions.

Definition 1 (ridesharing tour). *Let $S \subseteq P$ be a ridesharing group and let $V(S)$ denote the location set of participants in S, therefore, $V(S) = \cup_{i \in S}(s_i \cup t_i)$. A ridesharing tour for ridesharing group S, $R(S)$ is a directed Hamiltonian path on the graph $G(S) = (V(S), A(S))$ where $A(S) = V(S) \times V(S)$ such that*

1. *$R(S)$ starts from rider d's origin s_d and ends at d's destination t_d.*
2. *Let $S_{-d} = S \setminus \{d\}$. For every rider i in S_{-d}, s_i precedes t_i.*

Note that the above definition implies that rider d is assigned as the driver in the ridesharing group S.

Definition 2 (ridesharing partition). *A ridesharing partition $SP = \{S_1, \ldots, S_m\}$ is a set of ridesharing groups such that*

1. $\cup_{S_j \in SP} S_j = P$
2. $S_j \cap S_k = \emptyset \quad 1 \leq j \neq k$

Definition 3 (ridesharing plan). *A ridesharing plan for a ridesharing partition SP is a set of ridesharing tours $RP = \{R(S_j) \mid S_j \in SP\}$*

Define $f(RP)$ as the value of ridesharing plan RP that corresponds to a function f. Define the objective function of the ridesharing optimization problem as:

$$max\{f(RP)\}$$

In this paper, f is the accumulated values of all the ridesharing tours, which is defined by:

$$f(RP(SP)) = \sum_{S_j \in SP} V(R(S_j)) \tag{1}$$

Definition 4 (ridesharing optimization problem (RSP)). *An optimization problem of ridesharing is a 4-tuple $\langle I_Q, S_Q, f_Q, opt_Q \rangle$, where:*

- *I_Q: the set of the participants P and the corresponding graph $G = (V, A)$*
- *S_Q: the set of all ridesharing plans for all the ridesharing partitions of P*
- *f_Q: $f(RP)$ the value of ridesharing plan RP*
- *opt_Q: max.*

Cost and value of a shared-ride

Denote by $\mathcal{R}(S)$ the set of all the feasible ridesharing tours for S. Let $C(i)$ be the cost associated with each rider's trip. Let $C(i, 0)$ be rider i's cost for an individual trip without ridesharing. Let $C(i, R)$ be the cost of i for participating in a ridesharing tour R. The value of a shared-ride for a rider $i \in S$, $v_i(R)$, is defined as the cost associated with switching from driving a vehicle from origin to destination to participating in a ridesharing tour:

$$v_i(R) = C(i, 0) - C(i, R)$$

Naturally, $C(i, R) = C(i, 0)$ and $v_i(R) = 0$ if no participation occurs for i.

In a formed ridesharing tour $R \in \mathcal{R}(S)$, a rider i that participates in R will be assigned to be the designated driver $i = d$ and given the pick-up and drop-off sequence for all the other riders. Under the assumption that the major components of the cost are proportional to the driven mileage, including fuel cost and vehicle usage, and that potential time-related components are negligible or deemed by participating riders, we can approximate the cost C as the driven miles. We assume $C(i, 0) = dist(i)$, which indicates the direct distance traveled from i's origin to destination without any detour, and $C(i, R) = dist(R)$, which indicates the tour length of R for the designated driver d. Hence, $C(i, R) = 0$ for any non-driving participant in ridesharing group R. Each rider's value is defined as follows:

$$v_i(R) = \begin{cases} dist(i), & i \in S \setminus \{d\}, \\ dist(i) - dist(R), & i = d \end{cases}$$

The cumulative value of a ridesharing tour R is the summation of the values of all the riders for participating in the ridesharing.

$$W(R) = \sum_{i \in S} v_i(R) = \sum_{i \in S} dist(i) - dist(R)$$

Given a set of ridesharing group $S \subseteq P$, the ridesharing system decision-maker wants to maximize the cumulative value of $W(R)$. Since $dist(i)$ is a constant for every $i \in S$, this objective is equivalent to minimizing $dist(R)$, that is finding the optimal solution for the corresponding Hamiltonian path as for Definition 1. We call the optimal solution for this problem the optimal ridesharing tour of S and denote it as $C^*(S)$. The associated value of $C^*(S)$ is called the optimal value of S and is denoted by $W^*(S)$.

Now, suppose the given set of ridesharing participants P has formed m ridesharing groups. Let $SP = \{S_1, \ldots, S_m\}$ denote the set of these groups. Then we must have $\cup_{S_j \in SP} S_j = P$ and $S_j \cap S_k = \emptyset$ for every $1 \le j \ne k$. Note that SP is also known as a partition of P. We define the cumulative value of the participant set P under a partition SP as the sum of the optimal values of sets in SP, that is

$$W_{SP}(P) = \sum_{S_j \in SP} W^*(S_j)$$

A more ambitious objective aims to maximize the cumulative value of P, equivalent to find the optimal set partition SP^* such that

$$SP^* = \arg \max_{SP_i \in \mathcal{SP}(P)} W_{SP_i}(P)$$

We would like to emphasize that we are tackling the problem from a system's perspective, assuming individual ridesharing participants will accept whatever system's optimal solution offers, as long as it is compatible with their initial provided time and location constraints. A significant part of the ridesharing 'game' also includes the determination of an acceptable price and cost allocation associated with a given solution. In particular, the value gained by the system operating in a ridesharing mode must be shared fairly among participants. Riders will pay a 'fee', compatible with and proportional to the individual value gain. Drivers will instead receive a 'fee' compatible with and proportional to the individual value loss, which is their driven mileage. This is a non-trivial game-theoretic problem to be solved and is not dealt with in this paper, but a solution approach, based on the identification of the 'nucleolus', can be seen in Lu and Quadrifoglio (2019).

Modeling

Setting

Suppose we have n ridesharing participants $P = \{1, 2, \ldots, n\}$. Each of them has an origin location and a destination location. Denote the node set of origin and destination locations as V_O and V_D, respectively. Let node i be customer i's origin node $(1 \le i \le n)$ and $i + n$ be his destination node, then we have $V_O = \{1, 2, \ldots, n\}$ and $V_D = \{n+1, n+2, \ldots, 2n\}$.

We have a complete digraph $G_R = (V_R, A_R)$, where $V_R = V_O \cup V_D$ is the set of all nodes and $A_R = V_R \times V_R$ is the set of all edges. Let $C_R \in \mathcal{R}^{2n \times 2n}$ be the cost matrix with c^0_{ij} representing the travel cost from node i to node j.

Figure 1 provides an illustration showing two feasible solutions to the RSP on G_R. Note that the number in a node indicates its associated customer index. Nodes with a rectangular shape with a '+' label represent the destinations. Figure 1(a) is a solution that consists of purely individual trips. In Figure 1(b), customers 1 and 2 and customers 4, 5 and 6 form a ridesharing groups. The nodes in red belong to assigned drivers.

Transformation

For the formulation purpose, the RSP is transformed to the single-depot multiple traveling salesman problem with pickup and delivery constraints (SDMTSP-PD) in the following manner. Let $V_0 = \{0\}$ be a 'dummy' depot. The transformed graph is represented by $G = (V, A)$ where $V = V_R \cup V_0$ and A is the set of all the directed edges connecting any two vertices in V. The cost of the arcs in A is defined as

$$c_{ij} = \begin{cases} 0, & if i = 0 or j = 0, \\ c^0_{ij}, & otherwise. \end{cases}$$

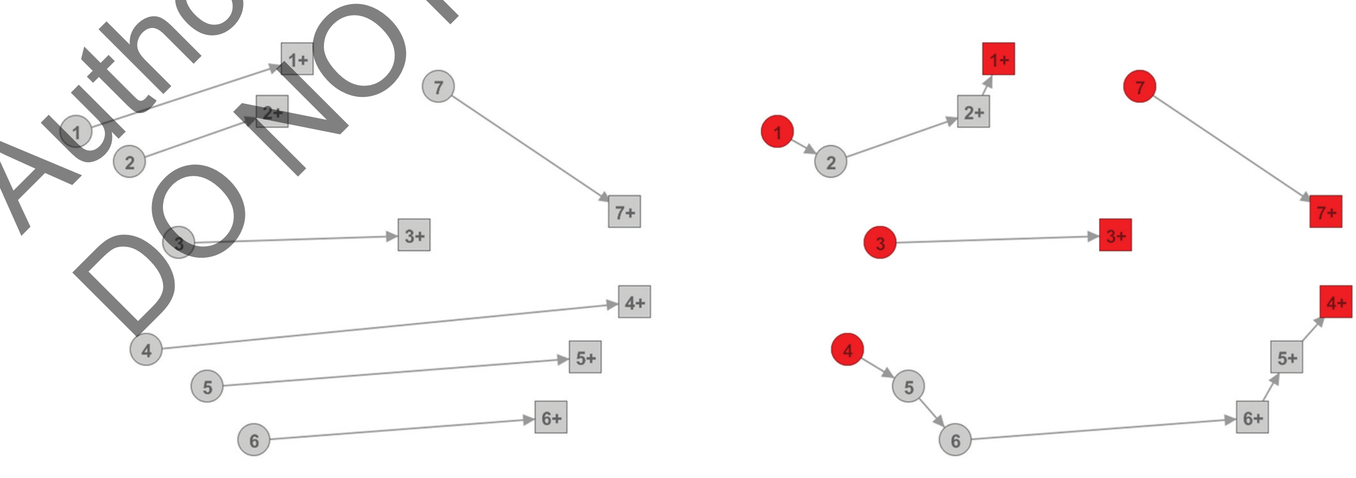
Here, the cost of the arcs is determined based on the distance, which is driven mileage. c^0_{ij}, which denotes the geometric distance between node i and node j, is introduced to differentiate from the formulation after the transformation using c_{ij}. c^0_{ij} does not include any extra costs associated with being a driver. The solution in Figure 1(b) is equivalent to the solution in Figure 2, where dash lines indicate zero-cost arcs.

Integer program

The corresponding integer program for the transformed SDMTSP-PD problem is defined as follows. For each edge (i, j) ∈ A, we define a binary variable x_{ij} such that

$$x_{ij} = \begin{cases} 1, & if (i, j) \in A \text{ is in the solution}, \\ 0, & otherwise. \end{cases}$$

Additionally, we define a binary variable y_{ik} as follows



(a) (b)

Figure 1. (a) Individual trips vs. (b) organized ridesharing trips.

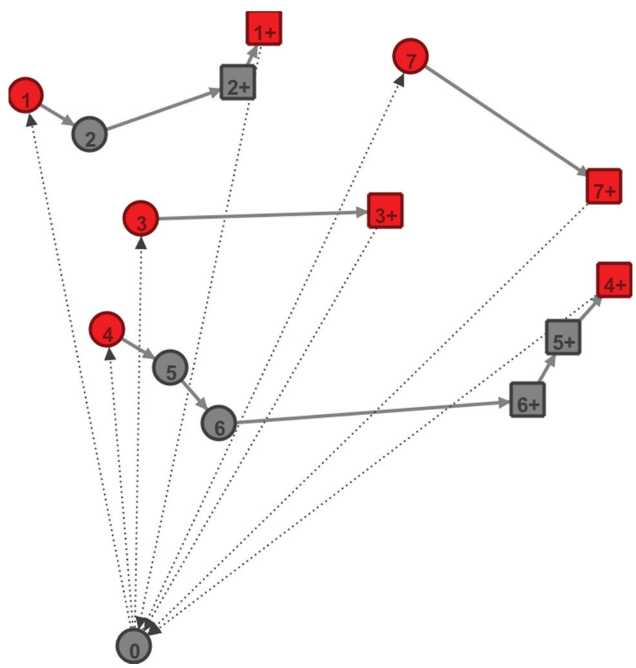


Figure 2. Corresponding feasible solution for SDMTSP-PD on the transformed graph.

$$y_{ik} = \begin{cases} 1, \textit{if node } i \textit{ is visited by driver } k, i \in V\backslash\{0\}, k \in V_o \\ 0, \textit{otherwise.} \end{cases}$$

Note that here $y_{ik} = 1$ implies customer k is a driver. u_i are continuous variables called node potentials that indicate the visit order of node i in the tour. p denotes the maximal number of nodes a driver can visit in a tour, which can be used to specify the seat capacity of participants' vehicles. In a typical dynamic ridesharing setting, most participants use private vehicles with a 5-seat capacity, which would need p to be set equal to 10.

$$minZ = \sum_{(i,j)\in A} c_{ij}x_{ij} \tag{2}$$

$$\sum_{i=0}^{2n} x_{ij} = 1, j = 1, 2, \ldots, n \tag{3}$$

$$\sum_{j=0}^{2n} x_{ij} = 1, i = 1, 2, \ldots, n \tag{4}$$

$$x_{0i} - x_{(i+n)0} = 0, i = 1, 2, \ldots, n \tag{5}$$

$$u_i - u_j + px_{ij} \leq p - 1, 1 \leq i \neq j \leq 2n \tag{6}$$

$$u_i \leq u_{i+n}, 1 = 1, 2, \ldots, n \tag{7}$$

$$\sum_k y_{ik} = 1, i = 1, \ldots, 2n; k = 1, \ldots, n \tag{8}$$

$$y_{ik} = y_{(i+n)k}, i = 1, \ldots, n; k = 1, \ldots, n \tag{9}$$

$$x_{0i} = y_{ii}, i = 1, \ldots, n \tag{10}$$

$$-\left(1 - x_{ij}\right) \leq y_{ik} - y_{jk} \leq \left(1 - x_{ij}\right), 1 \leq i \neq j \leq 2n; k = 1, \ldots, n \tag{11}$$

$$x_{ij} \in \{0, 1\}, 0 \leq i \neq j \leq 2n \tag{12}$$

$$y_{jk} \in \{0, 1\}, i = 1, \ldots, 2n; k = 1, \ldots, n \tag{13}$$

Constraints (3) and (4) are the continuity constraints. Constraints (5) make sure that a tour starts at its driver's origin and ends at his destination. Constraints (6) are a group of subtour-elimination constraints (SECs) first proposed by Miller, Tucker, and Zemlin (1960). Constraints (7) ensure that a customer's origin precedes his destination. Constraints (8) ensure that each node is visited by exactly one driver. Constraints (9) make sure that a customer's origin and destination are visited by the same driver. Constraints (10) mean that customer *i* is selected as a driver if and only if his origin is visited by himself. The intuitive meaning of constraints (11) is that if edge *(i, j)* is selected in the solution then nodes *i*, *j* must be served by the same driver. This set of constraints serve as a bridge between x variables and y variables. To improve computational efficiency, the following constraints that are entangled with the binary variable x_{ij} can be added to the program.

$$x_{0j} = 0, \quad j \in V_D \tag{14}$$

$$x_{i0} = 0, \quad i \in V_O \tag{15}$$

$$x_{i+n,i} = 0, \quad i \in V_O \tag{16}$$

$$x_{ij} = 0, \quad i = j \in V \tag{17}$$

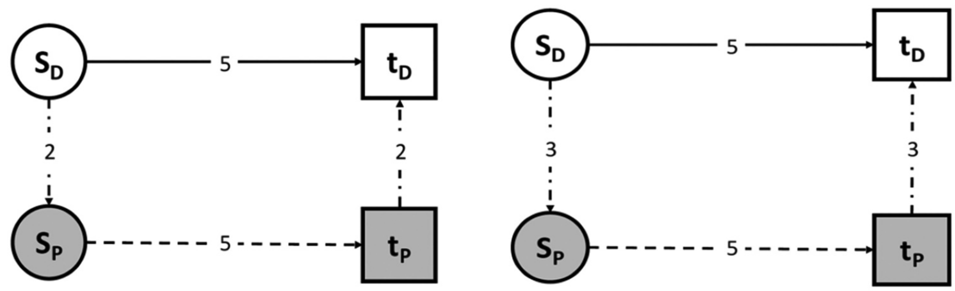
The problem is a generalization of the classic traveling salesman problem whose decision version belongs to the class of NP-complete (Karp 1972). The above formulation is not solvable in a reasonable time for large enough instances, such as those arising in a metropolitan area system-wide ridesharing service. Note that the above model, for a given pool of n participants, has a larger feasible region with more binary variables (namely $2n^2$ additional y_{ik} binary variable needed to select the drivers) making it much more challenging to solve compared to the more constrained version with predetermined drivers (in which all the y_{ik} are already set), even for relatively small sized instances. We also observe that the model could handle hybrid settings, with some participants preassigned as drivers/riders (just fix their corresponding y_{ik}) and the rest willing to be both as needed.

Solution methods

Profitability of ridesharing

Since the ridesharing problem aims to combine trips with similar itineraries together to save total travel cost, a natural question to ask is, when does ridesharing make the greatest economic sense, and thus should be encouraged, and in what case should inefficient ridesharing be discouraged. The following example illustrates this thought.

In Figure 3(a), the total travel cost, assuming no ridesharing happens, is $d_{S_D t_D} + d_{S_P t_P} = 10$. When *P* and *D* decide to share a ride, the optimal route yields a travel cost of 9 (e.g. $S_D - S_P - t_P - t_D$), resulting a cost saving equal to 1. In Figure 3(b) however, the optimal route has a travel cost of 11, resulting a cost loss equal to 1 instead of cost saving. Assuming again that the main components of the cost are proportional to the distance traveled, the profitability of ridesharing primarily depends on the relative geographical



(a) Profitable ridesharing (b) Nonprofitable ridesharing

Figure 3. A two-player example.

location of the participants. For a two-player ridesharing to be profitable, the combined route cost must be less than the sum of the two solo trip costs. That is, $d_{S_DS_P} + d_{t_Dt_P} + d_{S_Pt_P} < d_{S_Dt_D} + d_{S_Pt_P}$, i.e. $d_{S_DS_P} + d_{t_Dt_P} < d_{S_Dt_D}$. This is equivalent to say, the cost saving of ridesharing, $d_{S_Dt_D} - (d_{S_DS_P} + d_{t_Dt_P})$, has to be greater than 0. Of course, a more accurate analysis can be carried out considering a more granular definition of the cost; however, it would still possible to define a profitability associated with any match.

One-to-one match

Recall that the objective of minimizing total travel cost is equivalent to maximizing total cost saving. The above observation and the concept of profitability defined in Section 4.1 motivates Wang (2013) to solve an alternate version of ridesharing optimization problem: the optimal rideshare matching problem. The model also considers a single-rider single-driver rideshare matching with participants' flexible roles. In the optimal rideshare matching problem, each customer can be matched with at most another customer to form a shared-ride. Obviously, the solution to this problem is an upper bound of the solution to the ridesharing optimization problem. Nevertheless, the solution to this problem can provide insight in solving the ridesharing optimization problem and can serve as a benchmark to our proposed heuristic solution methods. It is important to emphasize that this matching approach is suitable specifically for our RSP without predetermined driver.

The optimal ridesharing matching problem can be modeled on a graph. Let $G = (V, A)$ be a digraph with $V = \{1, \ldots, n\}$ standing for the set of customers and $A = \{(i,j) \mid i,j \in V\}$ representing the possible rideshare match between agent i and j. A directed arc (i,j), $i,j \in V$ is associated with an edge cost c_{ij} equal to the cost saving when i serves as the driver and j as the rider in the $i-j$ rideshare match. The objective function of this problem aims to maximize the cost savings of rideshare matches over all possibilities: $\sum_{(i,j)\in A} c_{ij}x_{ij}$. Here x_{ij} is a binary decision variable that is defined as

$$x_{ij} = \begin{cases} 1, \; \mathit{if rideshare match}(i,j) \in A \mathit{is selected}, \\ 0, \mathit{otherwise}. \end{cases}$$

The constraints of this optimization model can be represented as follows:

$$\sum_{j\in V\setminus\{i\}} x_{ij} + \sum_{j\in V\setminus\{i\}} x_{ji} \leq 1 \quad i \in V$$

The problem is an instance of the Maximum Weight Match Problem, which has been first solved optimally in polynomial time ($O(n^3)$) by Edmonds (1965), later improved by an approximated linear time heuristic Duan and Pettie (2014). This is crucial as we can build on the optimal solutions of this Matching Problem, obtained in polynomial time also for larger instances, to improve our solution to the general RSP.

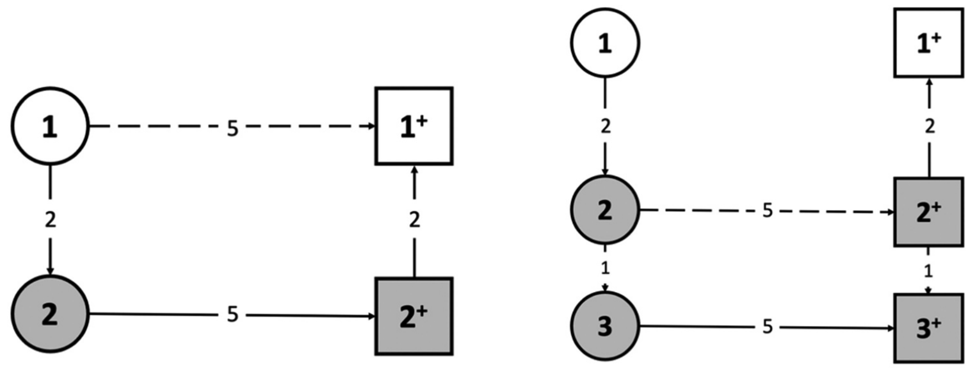
An insertion heuristic

The optimal solution to the Matching Problem found in the previous Section 4.2 will potentially include unmatched customers (those not able to find a feasible pairing match), still driving independently. However, the RSP allows 2+ ridesharing passengers. So, as long as the vehicle capacity is not reached, the route plan for the RSP could be further improved by adding unmatched customers to the existing plan, wherever convenient for the participants. Basically, the optimal Matching Problem solution is a feasible, good, but still very likely sub-optimal solution to the RSP. This section will describe a proposed methodology to find an improvement to the RSP starting from the optimal Matching Solution.

As an illustrative example of the concept, Figure 4(a) represents an optimal Matching Problem solution obtained by solving the optimization model in Section 4.2. This solution can be potentially further improved. As shown in Figure 4(b), when an unmatched customer 3 is available to participate, the total travel cost can be saved by including 3 into the existing route of 1 and 2. The saved cost by including traveler 3 is $d_{22+} - d_{23} - d_{2+3+} = 3$. Note that this addition would not be allowed in the Matching solution, as ridesharing passengers can be at most 2, but certainly appropriate for the RSP, when not violating the vehicle capacity.

The observation from Figure 4 motivates the authors to develop an insertion-based heuristic to improve the solution to the ridesharing optimization problem RSP. These routines are intuitive and effective in reaching near-optimal solutions in very short computational time (Quadrifoglio, Dessouky, and Palmer 2007), in particular, when starting from an already optimal Matching solution. However, many possible alternative construction heuristics can be applied to our base solution and potentially perform more efficiently.

The insertion-based heuristic algorithm is described in Algorithm 1 and it starts from the optimal matching solution (Section 4.2). For each unmatched participant, for which we are



(a) An optimal match (b) Insertion

Figure 4. A profitable insertion.

given the location, the algorithm loop through all the existing routes to find a feasible insertion position that has the greatest cost saving. Note that a feasible insertion may also modify the driver's role for an existing match, as an unmatched participant can be inserted prior to the original driver and drive to pick-up and drop-off both originally matched participants. This process repeats until either no unmatched participant is left or an insertion position with positive cost saving cannot be found. The output of the algorithm is the improved ridesharing solution, which is a route plan for the RSP. As the insertion heuristics are known to have a quadratic complexity, the overall complexity of the procedure is dominated by the Matching portion ($O(n^3)$).

Experiments

We implemented the algorithms in Java with CPLEX 20.1 and the Concert library. We ran experiments for up to 1 day (86,400 sec) using an i7-8750H CPU @2.2GHz with 6 Cores. The data set[1] we used in the experiments were selected from Dumitrescu et al. (2010), in which the origins and destinations of participants were randomly generated in the square [0, 1000] × [0, 1000]. The Euclidean distances were used. Vehicles' maximum capacity is set to 5 (p = 10). The instances are named prob*n*X, where *n* is the participant size, and X stands for different instances with the same size. The experimental results are summarized in Table 1, where costs are shown. 'Solo' is the solution without any ridesharing (all passengers driving individually). 'Match' is the optimal matching solution found in Section 4.2. 'Ins' stands for insertion and is the solution to the RSP found with the proposed insertion heuristic procedure of Algorithm 1. 'MIP' is the optimal solution to the RSP found with the CPLEX solver and its solving time (* means optimal RSP). Then the 'Gap' is shown between Ins and MIP solutions. The last column 'P-RSP' shows optimal (*) costs (for each instance with at least 10 participants) for the typical ridesharing service with arbitrarily predetermined drivers. Roughly half of the participants are assigned as drivers (simply the first rounded half based on their order shown in each data set), as we observed that most general RSP solutions eventually select approximately half participants as drivers. Note that, by definition, any optimal P-RSP is bounded by its corresponding general optimal RSP.

Algorithm 1: An insertion algorithm based on the optimal matching solution

```
input: Geolocations of customers and the optimal match solution
output: An improved ridesharing solution

SAVING=0 ;
while true do
    if UNMATHCED = φ then
        break ;
    end if
    SOLUTION = φ ;
    SAVING = 0 ;
    for a ∈ UNMATCHED do
        for r ∈ all_routes do
            for each feasible insertion position do
                Δ = costsavingafterinsertion ;
                if Δ < SAVING then
                    SAVING = Δ ;
                    record the incumbent solution ;
                end if
            end for
        end for
    end for
    if SAVING < 0 then
        Update UNMATCHED ;
        Update all_routes ;
        Update SOLUTION ;
    end if
    else
        break ;
    end if
end while
```

In all instances, both the optimal matching and the insertion heuristic are able to find the solutions instantly (in less than 1 second). Recall that the RSP is not only NP-hard, but greatly more complex than its special case with predetermined drivers (P-RSP), as it is required to optimally find values for $2n^2$ additional y_{ik} binary variables needed to select the drivers. Therefore, even a relatively small instance will require significantly more time to be optimally solved compared to any P-RSP. Instances up to 10 participants are solved to optimality (even though some with significant CPU time).

As an illustrative example, Prob10e has the following solutions ('+' means drop-off):

- Solo (cost = 5303): all solo drivers
- Match (4965): 2/4/6/7/9/10 unmatched solo drivers

Table 1. Experiment results.

prob n X	Total cost Solo	Match	Ins	MIP	[Time(s)]	Gap% Ins-MIP	P-RSP
5a	2722	2338	2338	2338*	[< 1]	0.0%	
5b	2378	2115	2115	2115*	[< 1]	0.0%	
5c	3189	2856	2663	2663*	[< 1]	0.0%	
5d	2086	1842	1842	1842*	[< 1]	0.0%	
5e	2171	2171	2171	2171*	[< 1]	0.0%	
10a	6110	4681	4681	4267*	[3997]	9.7%	5102*
10b	5577	4966	4618	4487*	[24928]	2.9%	4986*
10c	5514	4109	3592	3592*	[502]	0.0%	4308*
10d	4126	3662	3662	3604*	[239]	1.6%	3691*
10e	5303	4965	4810	4545*	[1979]	5.8%	5108*
15a	6494	5633	5569	5112	[86400]	8.9%	5755*
20b	10131	8233	8048	7305	[86400]	10.2%	8861*
25a	11781	10053	9790	8982	[86400]	9.0%	10480*
30a	17112	13366	11849	11469	[86400]	3.3%	14877*
35b	16051	13136	11799	11484	[86400]	2.7%	14677*

- driver 3 → (3)-(1)-(1+)-(3+)
- driver 8 → (8)-(5)-(5+)-(8+)

• Ins (4810): 2/6/10 unmatched solo drivers

- driver 7 → (7)-(3)-(1)-(1+)-(3+)-(7+)
- driver 8 → (8)-(4)-(9)-(4+)-(5)-(5+)-(9+)-(8+)

• MIP (4545*, optimal): 2/4/6/7 unmatched solo drivers

- driver 8 → (8)-(5)-(5+)-(8+)
- driver 9 → (9)-(3)-(1)-(3+)-(1+)-(10)-(10+)-(9+)

Larger instances are solved starting from the Ins feasible solution to look for any potential improvement for up to 86400 sec (1 day). It is noteworthy to point out that the solution obtained from the insertion heuristic is close to the solutions obtained via solving MIP: when not optimal, the gap between 'Ins' and 'MIP' ranged from 1.6% to 10.2%. The optimal costs for the sampled P-RSP are all reached within the 86400 sec and are better than Solo costs, as expected, but worse than RSP costs, solved optimally (MIP) or heuristically (Ins). This confirms that the RSP provides a better system-wide solution than P-RSP, even if the larger instances are not yet guaranteeing RSP optimality, but are well approximated by our heuristic, whose values are also better than the optimally solved selected P-RSP instance.

Table 2. Performance of algorithms – saving cost.

Problem size	Total cost (saving%) Solo	Match		Insertion	
5	2509.5	2264.5	(9.8%)	2225.9	(11.3%)
10	5325.9	4476.5	(15.9%)	4272.7	(19.8%)
15	7929.6	6654.7	(16.1%)	6499.7	(18.0%)
20	10561.8	8454.6	(19.9%)	8201.7	(22.4%)
25	12695.5	10430.8	(17.8%)	9826.4	(22.6%)
30	16490.1	12975.2	(21.3%)	12190.0	(26.1%)
35	18367.0	14327.1	(22.0%)	13576.6	(26.1%)

To further compare the solution qualities of insertion heuristic (Ins) and optimal matching (Match) in terms of cost-saving, the solution values (average of five instances for each size) are summarized in Table 2. The percentage in parenthesis for each solution method indicates the cost-saving percentage compared to the non-ridesharing route plan (Solo). It can be seen that Insertion can further save about 5% of the total travel cost compared to the Match starting solution.

Figure 5 presents 95% confidence intervals for the values shown in Table 2. We note that variability is quite limited, even if only 5 instances per case were included in the calculation. The figure also shows that the difference between Solo and the other two becomes more significant as the problem size increases.

Note that average detour factor for the passengers participating into the ridesharing game is 1.3 (30% over their original direct Solo

Table 3. Performance of algorithms – saving vehicles.

Problem size	Vehicle trips (saving%) Solo	Match		Insertion	
5	5	4.0	(20.0%)	3.8	(24.0%)
10	10	7.2	(28.0%)	5.8	(42.0%)
15	15	9.8	(34.7%)	8.2	(45.3%)
20	20	12.2	(39.0%)	9.8	(51.0%)
25	25	16.2	(35.2%)	11.6	(53.6%)
30	30	18.2	(39.3%)	13.8	(54.0%)
35	35	21.8	(37.7%)	15.8	(54.9%)

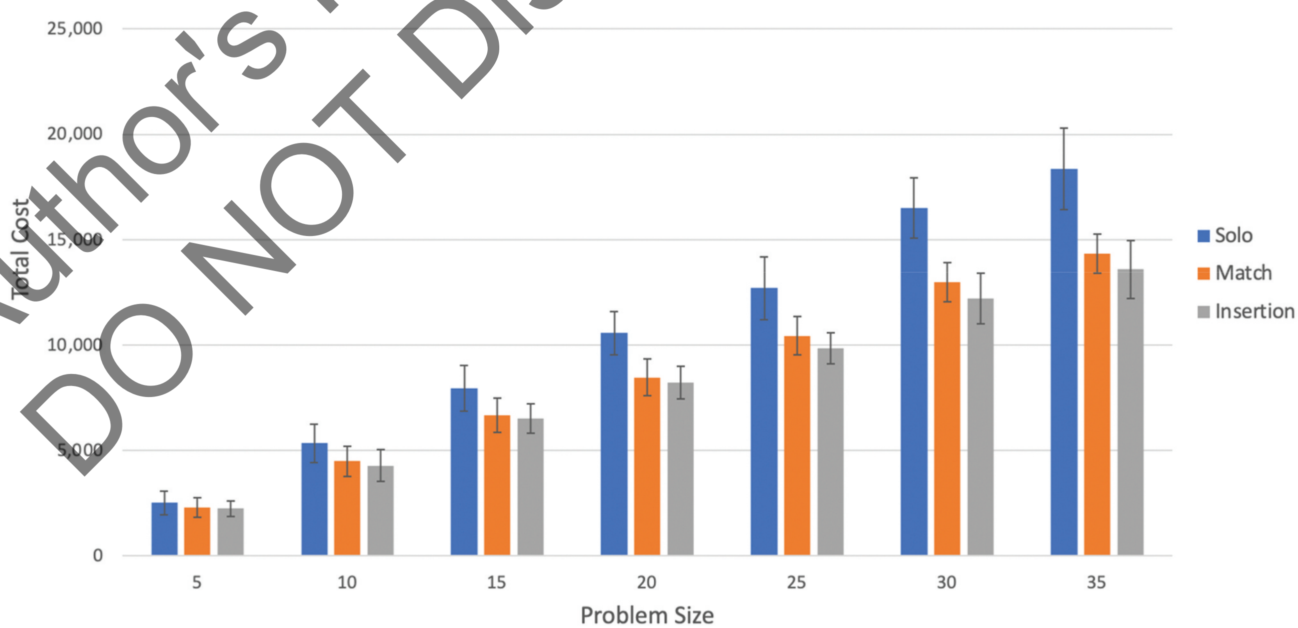


Figure 5. Total costs with 95% C.I.

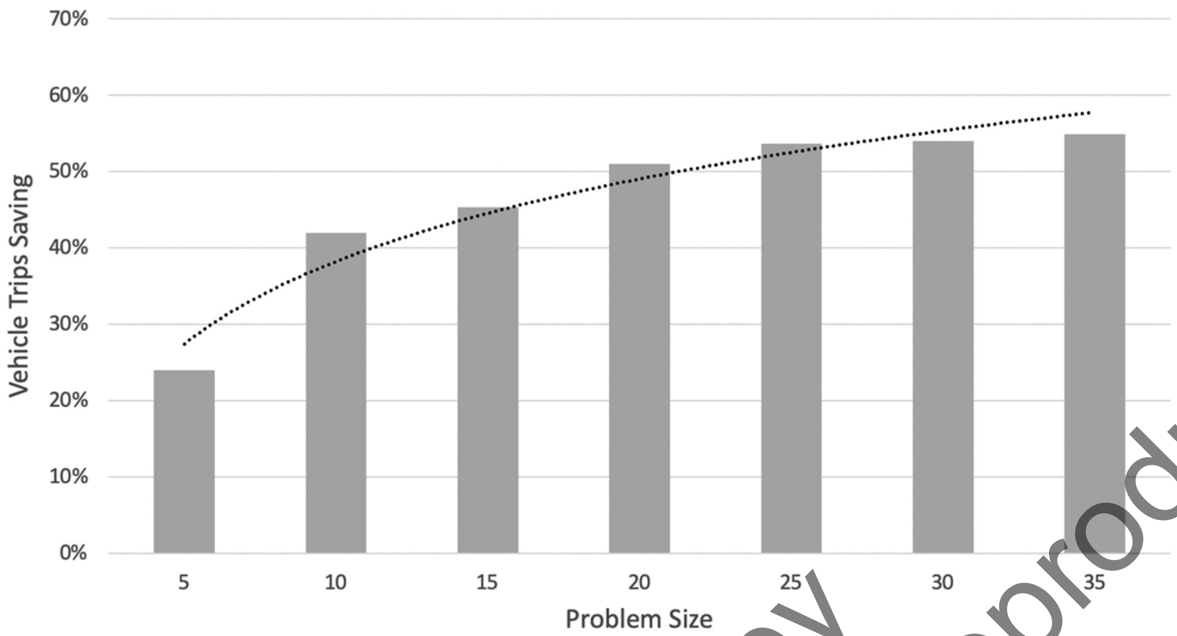


Figure 6. Cost saving vs. Problem size.

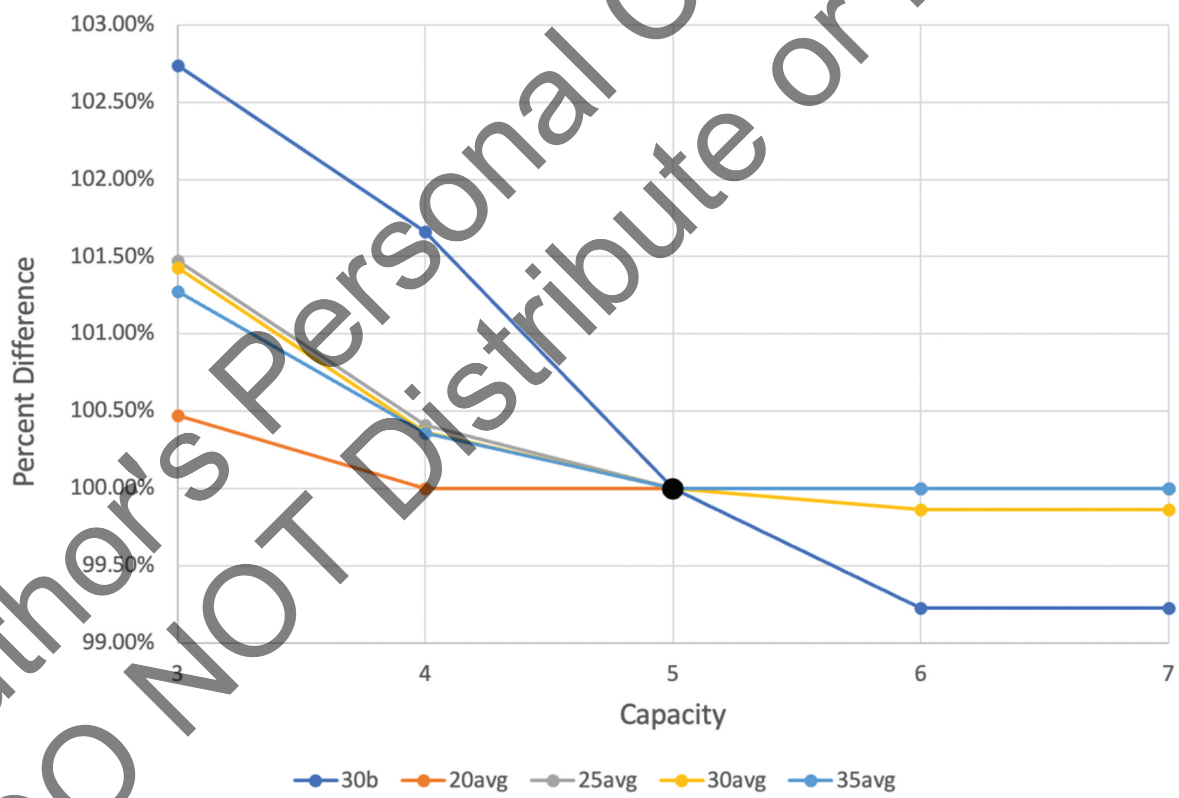


Figure 7. Cost sensitivity vs. Capacity.

ride). A minor portion (about 3%) is experiencing detours factors higher than 2, which might be deemed an inappropriate service level in a practical implementation (even though participants are made aware of the rules beforehand). This may be prevented by simply adding proper constraints in the MIP model and the heuristic without significantly altering our current results.

Table 3 summarizes the saved vehicle trips by adopting ridesharing. As can be seen, Match and Insertion can save 20–38% and 24–55% vehicle trips on road, respectively, depending on the problem size. Once again, Insertion improves Match significantly (as high as 18%). Also, note that the vehicle trip indicates average number of trips saved and the percentage of vehicle trips saving increases as the problem size increases, which confirms the scale effect of ridesharing. Figure 6 depicts the percentage of vehicle trips saved by Insertion as shown in the last column of Table 3 with the logarithmic trendline with R^2 of 0.9427, which explains the scale effect. The trendline shows the increment of the percentage of vehicle trips saving with increasing the problem sizes. Also, once the problem size gets bigger than 20, the level of vehicle trip saving would be greater than 50% compared to the non-ridesharing plan.

Figure 7 shows a sensitivity analysis over the most significant parameter of our problem, which is the vehicle capacity C. As the base case has been considered C=5 (to match most actual private vehicles), the average percentage difference in total costs is shown for 20, 25, 30 and 35 participants (20avg, 25avg, 30avg, 35avg). The total costs increase as capacity becomes lower and vice-versa. However, the sensitivity appears low and is even less significant for C > 5. This is an indication that vehicles are rarely filled up to capacity. In addition to the average values, we also show our most sensitive case (prob30b), showing instead a more evident dependency on capacity changes.

Conclusions

This paper considers the most generalized ridesharing problem, identified by a set of time-compatible participants, among which the driver/rider roles are not predetermined, but left to decide by the system to further reduce costs compared to the typical version with predetermined drivers. Although large-scale ridesharing service providers may find themselves in need to face this challenging problem increasingly often, it has not been studied in the literature from this specific perspective. Notably, given the origin and destination of participants, how should the service provider organize the service by assigning the driver/rider roles and suggesting a route to minimize system-wide travel cost? We provide a formal definition of this problem as the generalized ridesharing optimization problem (RSP) and showed how to transform it to the single-depot multiple traveling salesman problem with pickup and delivery constraints (SDMTSP-PD) and with capacity constraint, for which a mixed-integer problem (MIP) model was then developed.

The problem is NP-hard and much more challenging than, for example, the typical TSP problem with the same number of participants, as driver's role binary decision variables are left to decide by the system. Thus, we resort to an approximation procedure to solve larger instances. In particular, optimal matching is suitable for the problem's general structure without predetermined drivers. It is pivotal as it is solved in polynomial time and used as a good base solution to be improved with an insertion-based heuristic. The approximated solution was then compared to the optimal solution of RSP for optimally solvable instances. We note that the experiments conducted in this paper were using randomly generated geographical locations. But in a real-world scenario, travelers' origins and destinations are more likely to be clustered. In these situations, heuristics can potentially perform even better, as the proposed formulation has a robust structure allowing clusters to be solved in a parallel algorithm fashion.

Our results suggest the following managerial consideration for such systems: (1) Significant additional cost savings are achieved by relaxing the predetermined drivers' role

among willing participants. Incentives should be considered to foster such configuration. (2) Near optimal solutions of the very complex system can be achieved by a heuristic approach, such as the one proposed. (3) A progressive scale effect is evident, the percentage of vehicle trip-saving increases as more customers join the ridesharing service; so larger systems are more efficient. (4) Capacity is important but not critical, so smaller vehicles would suffice and also be more efficient and flexible.

While our results are deemed good for practical purposes, other and better solutions approaches are certainly possible. In particular, the special structure of the RSP MIP model is still to be investigated and exploited, thus developing valid inequalities and logic cuts is a promising direction. Lastly, the formulated framework with a few adjustments potentially matches a fleet of automatic vehicles serving passengers, likely the future transportation solution for ridesharing services.

Note

1. The data sets can be downloaded from http://www.diku.dk/~sropke/

Acknowledgments

The research reported in this paper was partially supported by a Dissertation Fellowship of Texas A&M University.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

The author(s) reported there is no funding associated with the work featured in this article.

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